Reliability of the Optimized Perturbation Theory for scalar fields at finite temperature

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Abstract. The thermodynamics of a massless scalar field with a quartic interaction is studied up to third order in the Optimized Perturbation Theory (OPT) method. A comparison with other nonperturbative approaches is performed such that the reliability of OPT is accessed.

Quantum field theory at finite temperature is the natural scenario to deal with phase transitions and with thermodynamic properties of equilibrium states. The range of applications goes from the understanding of the early universe to the low-energy effective theories in particle physics and condensed matter systems. Nevertheless, this sort of phenomena face a major complication, the break down of perturbation theory, caused by large fluctuations that can emerge in the systems due to infrared divergences and close to the critical points. Therefore, nonperturbative methods are required in general. In this work we study the thermodynamics of a massless scalar field with a quartic interaction in the context of the nonperturbative method of the Optimized Perturbation Theory (OPT). We evaluated the free energy up to the to third order in the OPT formalism and carried out the calculation of the pressure. The reliability of our results for this model is discussed and we also make a comparison with another nonperturbative approach, the Screened Perturbation Theory (SPT).

We consider a massless scalar field theory with $g\phi^4$ interaction and with Lagrangian density given by

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{g^2}{4!}\phi^4.$$ (1)

The implementation of the OPT $\mathscr{L}$ is performed through a linear interpolation,

$$\mathscr{L} \to \mathscr{L}^\delta = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{\delta g^2}{4!}\phi^4 - (1-\delta)\eta^2\phi^2 + \Delta\mathscr{L}_{\text{ct}}^\delta,$$ (2)

where $\Delta\mathscr{L}_{\text{ct}}^\delta$ contains the renormalization counterterms required to render the theory finite. In $\mathscr{L}^\delta$, δ is a dimensionless bookkeeping parameter used only to keep track of the order that the OPT is implemented (it is set equal to one at the end) and η is a (mass) parameter determined variationally at any giver order of the OPT. We choose the variational criterion known as the Principle of Minimal Sensitive (PMS), defined by

$$\left.\frac{d\Phi^{(k)}}{d\eta}\right|_{\bar{\eta},\delta=1} = 0\,,$$ (3)

which is applied to some physical quantity $\Phi^{(k)}$ calculated up to some k-order in the OPT. The optimum value $\bar{\eta}$ which satisfies Eq. (3) is a nontrivial function of the couplings (and of the original parameters of the theory), leading to the generation of nonperturbative results. The interpolation introduces only quadratic terms, which implies that the renormalizability is preserved [1, 2].

SPT is simply a reorganization of the perturbative series for thermal field theory [3]. The Lagrangian density in SPT is written as

$$\mathcal{L}_{\text{SPT}} = -\mathcal{E}_0 + \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2}(m^2 - m_1^2)\phi^2 - \frac{g^2}{4!}\phi^4 + \Delta\mathcal{L} + \Delta\mathcal{L}_{\text{SPT}}, \quad (4)$$

where $\mathcal{E}_0$ is the vacuum density term, and a mass term is added and subtracted. Setting $\mathcal{E}_0 = 0$ and $m_1^2 = m^2$, the original Lagrangian is recovered. SPT is defined by taking m^2 to be of order unity and m_1^2 to be of order g^2, expanding systematically in powers of g^2 and setting $m_1^2 = m^2$ at the end of the calculation. New ultraviolet divergences are generate in SPT, but they can be canceled by the additional counterterms in $\mathcal{L}_{\text{SPT}}$.

The mass parameter m in SPT is completely arbitrary. In order to complete a calculation using SPT, a prescription for the mass parameter m as a function of g and T is needed. A discussion for different possibilities for this prescription is made in [4]. We restrict ourselves to the tadpole gap equation,

$$m_t^2 = g^2 \frac{\partial \mathcal{F}}{\partial m^2}\bigg|_{m_1 = m}. \quad (5)$$

FIGURE 1. Contributions to the effective potential at $\mathcal{O}(\delta)$, F_1.
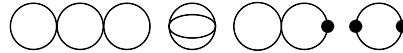
**FIGURE 2.** Contributions to the effective potential at $\mathcal{O}(\delta^2)$, F_2.
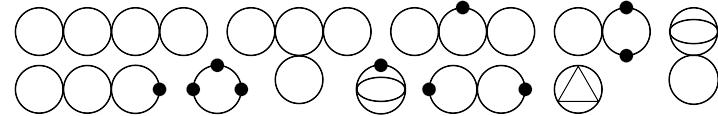
FIGURE 3. Contributions to the effective potential at $\mathcal{O}(\delta^3)$, F_3.

We apply the PMS, Eq. (3), directly to the free energy, which in terms of Feynman diagrams in the OPT formalism at $\mathcal{O}\left(\delta^3\right)$ reads

$$\mathcal{F} = F_0 + F_1 + F_2 + F_3, \quad (6)$$

where F_k is the free energy evaluated up to order k in OPT and shown in Figs. 1-3. The contributions shown were calculated in the SPT context through four-loops expanding in a double power expansion in m/T and g^2 and truncating at order g^7 [4]. The expansion required the evaluation of a nontrivial three-loop diagram, given by the second term in Fig. 2, which can be done using the techniques developed in [5].

We are now in position to determine the normalized pressure, which is given by $\mathscr{P}/\mathscr{P}_{\text{ideal}} = -\mathscr{F}$, where $\mathscr{P}_{\text{ideal}} = \pi^2 T^4/90$ is the pressure of an ideal gas of massless particles. The numerical results obtained from the application of the optimization procedure (PMS) in the OPT are shown in Figs. 4. We compare our results with those obtained from the application of the tadpole gap equation at the free energy up to 4-loops in the SPT formalism.
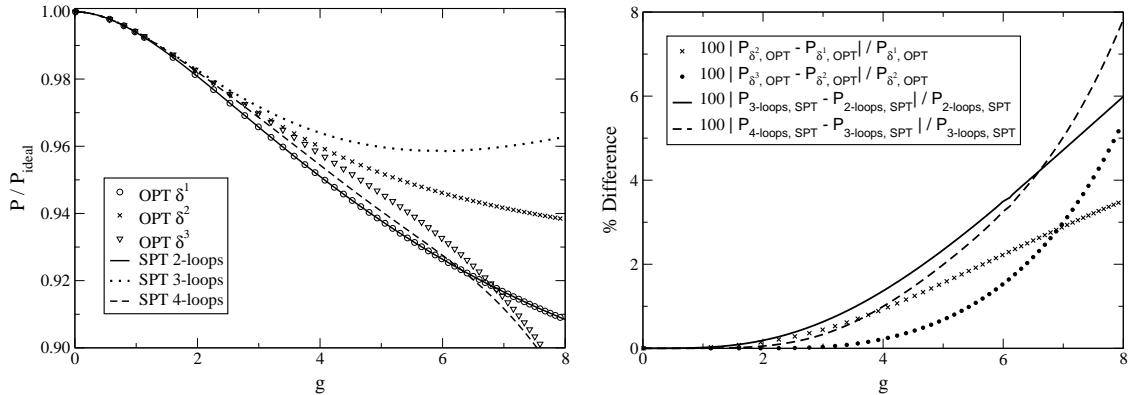

FIGURE 4. Normalized pressure in OPT and SPT approaches (left) and the percentage difference between the results (right).

As shown in Fig. 4, the percentage difference between P_{δ^3} and P_{δ^2} seems to indicate a convergence in the method faster than SPT. We expect to be able, using OPT up to $\mathscr{O}(\delta^3)$, to evaluate thermodynamical quantities of interest. Our results suggests a better reliability of the OPT method.

A difference between the methods that appears to be crucial for the improved results with OPT is the optimization procedure. In the SPT, the pressure through k-loops is calculated by using the solution to the $(k-1)$-loop tadpole equation, while in the OPT, we use the solution of the PMS applied to the k-order free energy to evaluate the corresponding k-order pressure.

ACKNOWLEDGMENTS

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